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NOTE

A NOTE ON "AUTOMORPHISM GROUPS OF PARTIAL ORDERS"

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1. Background

We refer to articles by Bird [1] and Bird et al. [2] on automorphisms of posets. Let P, Q denote posets; $P\times Q$ is the cartesian product with the lexicographic order and $P\otimes Q$ that same product with the "reverse" lexicographic order, viz. $(p, q)<(p', q')$ iff $q<q'$ or $q=q'$ and $p<p'$. $\Gamma(P)$ denotes the automorphism group of P, i.e. all order preserving maps of P onto P which have order preserving inverses. The wreath product $\Gamma(Q)\wr\Gamma(P)$ is the group of permutations of the cartesian product of P with Q given by pairs (b, f) with $b\in\Gamma(P)$ and $f\in(\Gamma(Q))^P$ which act on $(p, q)\in P\times Q$ by $(b, f)(p, q)=(b(p), f_p(q))$. The terminology and notation is essentially that found in [2], where the following result is established:

Theorem 1.1. *$\Gamma(Q)\wr\Gamma(P)\subseteq\Gamma(P\times Q)$ and we have $\Gamma(Q)\wr\Gamma(P)=\Gamma(P\times Q)$ iff each $g\in\Gamma(P\times Q)$ acts uniformly in the first coordinate (i.e. $g(p, q)$ and $g(p', r)$ have the same first coordinate whenever $p=p'$.)*

A similar result found in [1] replaces equality with isomorphism in a dual statement about the reverse lexicographic order:

Theorem 1.2. *$\Gamma(P)\wr\Gamma(Q)\cong\Gamma(P\otimes Q)$ iff each $g\in\Gamma(P\otimes Q)$ acts uniformly in the second coordinate (i.e. $g(p, q)$ and $g(p', q')$ have the same second coordinate whenever $q=q'$.)*

The proof of Theorem 1.2 found in [1], however, shows only that the natural imbedding is an isomorphism precisely under the stated conditions. One may well inquire whether other isomorphisms might not in fact exist, even though the stated uniform condition is not met. For P, Q finite Theorem 1.2 is equivalent to Theorem 1.1, however, the general assertion in Theorem 1.2 is false as the following counterexample shows.

2. An example

Let $P = 1 \oplus 2$ and $Q = \oplus \sum_{1 \leq n \in \omega} (n \oplus n)$ where ordinals $n \in \omega$ are as usual chains of type n; $\oplus$ denotes disjoint union of posets (see Fig. 1).

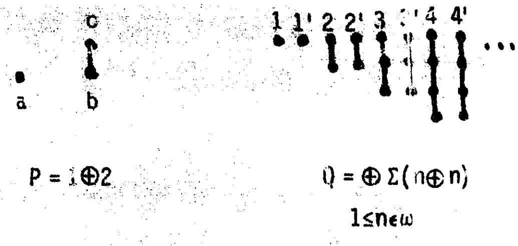

Fig. 1. Posets "P" and "Q".

Clearly $\Gamma(P) \cong \{\mathrm{id}\}$ and $\Gamma(Q) \cong (c_2 \times c_2 \times \cdots) =$ direct product of ω copies of c_2 the cyclic group of order 2. Thus $\Gamma(P) \wr \Gamma(Q) = \mathrm{id} \wr (c_2 \times c_2 \times \cdots) \cong (c_2 \times c_2 \times \cdots) =$ direct product of ω copies of c_2. But $P \otimes Q$ may be thought of as Q copies of P ordered as Q is ordered (see Fig. 2). Thus $\Gamma(P \otimes Q) \cong (\Gamma(1 \oplus 1) \times \Gamma(2 \oplus 2)) \times \Gamma(4 \oplus 4) \times \Gamma(6 \oplus 6) \times \cdots \times \Gamma(2n \oplus 2n) \times \cdots$.

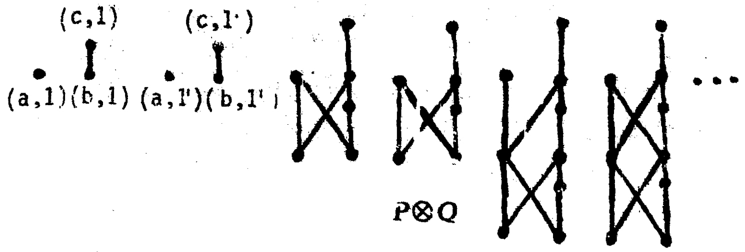

Fig. 2. The "Reverse" lexicographic order on $P \times Q$.

Since the only automorphisms exchange like connected components of $P \otimes Q$. Thus $\Gamma(P \otimes Q) \cong (c_2 \times c_2 \times \cdots) =$ direct product of ω copies of c_2, and we have $\Gamma(P) \wr \Gamma(Q) \cong \Gamma(P \otimes Q)$. However, $\Gamma(P \otimes Q)$ fails to act uniformly in the second coordinate: the map α which exchanges $(a, 1)$ with $(a, 1')$ and leaves everything else fixed belongs to $\Gamma(P \otimes Q)$ but we have $\alpha(a, 1) = (a, 1')$ and $\alpha(b, 1) = (b, 1)$ with $1 \neq 1'$. Thus Theorem 1.2 fails for infinite posets P, Q.

References

[1] E. H. Bird, Automorphism groups of partial orders, Bull. Am. Math. Soc. 79 (1973) 1011–1015.
[2] E. H. Bird, C. Greene and D. J. Kleitman, Automorphisms of lexicographic products, Discrete Math. 11 (1975) 191–198.